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FLAGIFIED BOTT MANIFOLDS AND THEIR MAXIMAL TORUS ACTIONS

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1. Introduction –Why flagified Bott manifolds?–

This article is a research announcement of the part of the results in the progress paper [KLSS] about geometry, combinatorics and algebra (representation theory), so-called *Bott's triangle*, of *flagified Bott manifolds.*

In this paper, we introduce the following theorem.

Theorem 1.1. *The natural torus action on a flagified Bott manifold is the maximal torus action preserving its complex structure.*

We first introduce the motivation of flagified Bott manifolds. A Bott manifold, defined by Grossberg-Karshon in [GK], is "topologically" obtained as the iteration of complex projectivization of Whitney sum of two complex line bundles. More precisely, a *Bott manifold* B_n is a sequence

$$B_n \to B_{n-1} \to \cdots \to B_1 \to B_0 = \{pt\}$$

of manifolds $B_j = \mathbb{P}(E_j)$, where E_j is a Whitney sum of two complex line bundles over B_{j-1} and $\mathbb{P}(E_j)$ is its projectivization, i.e., B_n is diffeomorphic to the iterated $\mathbb{C}P^1$-bundles. We call $\{B_j \mid j = 0, \ldots, n\}$ a *Bott tower* of height n (or an *n-stage Bott tower*). On the other hand, Grossberg-Karshon also introduce the complex manifold by using the notion of "Lie theory", called a *Bott-Samelson variety.* Here, a *Bott-Samelson variety* is defined as follows. Let G be $SL(r+1, \mathbb{C})$, and B be its Borel subgroup. Let W be the Weyl group of G, which is isomorphic to the symmetric group S_{r+1}. Fix the generators (simple reflections) of $W(\simeq S_{r+1})$, and choose some s_j, $j = 1, \ldots, n$, from the generators. The following submanifold $X(s_1, \ldots, s_n) \subset (G/B)^n$, determined by these generators, is called a *Bott-Samelson variety*:

$$\begin{aligned} &X(s_1, \ldots, s_n) \\ &= \{(g_1B, g_2B, \ldots, g_nB) \mid g_{j-1}^{-1}g_j \in \overline{Bs_jB}, \text{ for all } j = 1, \ldots, n\}, \end{aligned}$$

where $g_0 = e$. A Bott manifold is also obtained by deforming the complex structure of a Bott-Samelson variety. Using this deformation, Grossberg-Karshon studies the interesting relations among geometry, combinatorics and algebra of Bott manifolds, so-called *Bott's triangle*.

Apart from the original motivation, toric topologists generalize Bott manifolds from topological point of view in [CMS, KS14, KS15] as an effective testing ground for cohomological rigidity problems. Their definitions are just changed the assumptions of complex vector bundles E_j in the (topological) definition of Bott manifolds. Namely, if we change the assumption of each E_j to any Whitney sum of complex line bundles then B_n is called a *generalized Bott manifolds* in [CMS], and if we change that to any complex vector bundles then B_n is called a *complex projective towers* (or $\mathbb{C}P$-*tower*) in [KS14, KS15]. Both manifolds are diffeomorphic to the iterated complex projective bundles. However, it seems to be hard to find the (natural) counterparts corresponding to Bott-Samelson variety for these topological generalizations. Therefore, from the point of view of studying Bott's triangle, these topological generalizations are not so effective. So in this paper, we introduce more natural generalization of Bott manifolds (from the point of view of studying Bott's triangle).

In our definition in Section 2, the obtained topological space is the iterated flag bundles instead of the iterated complex projective bundles, so we call it a *flagified Bott manifold.* We also define a *flagified Bott-Samelson variety* as a generalization of a Bott-Samelson variety. Moreover, we show that the flagified Bott manifold has the structure of a GKM manifold in the sense of Guilleminn-Zara [GZ]; however, a flagified Bott-Samelson variety with the natural torus action does not have the structure of a GKM manifold though they are diffeomorphic. In Section 3, we give an outline of the proof of Theorem 1.1 by using the invariant defined in the paper [Ku].

2. Flagified Bott manifolds

We first introduce the definition of *flagified Bott manifolds.* To do that, we prepare the general notations. Let F be a (smooth, compact, connected) manifold and E be a complex vector bundle over F. Let $E_p \subset E$ be the fibre over $p \in F$. The *associated flag bundle* $Flag(E) \to F$ is obtained from E by replacing each fiber E_p by the flag manifold $Flag(E_p)$, i.e., the set of full flags $\{0\} \subset V_1 \subset V_2 \cdots \subset V_n = E_p$ in E_p, where V_i is a complex i-dimensional vector subspace in $E_p(\simeq \mathbb{C}^n)$ (Cf. the definition of projectivization of complex vector bundles).

Definition 2.1. A *flagified Bott tower* $\{F_j \mid j = 0, \dots, m\}$ of height m (or an *m-stage flagified Bott tower*) is a sequence:

$$F_m \to F_{m-1} \to \cdots \to F_1 \to F_0 = \{pt\},$$

of manifolds $F_j = Flag(E_j)$, where E_j is a complex vector bundle over F_{j-1} which splits into complex line bundles (we often denote $E_j = \oplus_{i=1}^{n_j+1} \xi_{i,j}$ by a complex line bundle $\xi_{i,j}$ over F_{j-1}). We call F_j the *j-stage flagified Bott manifold* of the flagified Bott tower. In particular, we call F_m a *flagified Bott manifold.*

2.1. **Properties of flagified Bott manifolds.** Because each line bundle over F_{j-1} is a torus equivariant bundle and the flag manifold has the natural torus action, there is an $(n_1 + \cdots + n_m)$-dimensional torus action on F_m which preserves the complex structure. Moreover, by definition, it is easy to check that the flagified Bott manifold F_m is equivariantly diffeomorphic to the following twisted product:

$$\begin{aligned} F_m &\cong \prod_{j=1}^{m} PU(n_j + 1) / \prod_{j=1}^{m} T^{n_j} \\ &:= PU(n_1 + 1) \times_{T^{n_1}} (PU(n_2 + 1) \times_{T^{n_2}} (\cdots \times PU(n_m + 1)/T^{n_m}) \cdots), \end{aligned}$$

where $PU(n+1) \simeq SU(n+1)/\mathbb{Z}_{n+1}$, i.e., the quotient group of $SU(n+1)$ by its center, $T^{n_j} \subset PU(n_j + 1)$ is the maximal torus and T^{n_j}-action on the latter factors, i.e., the action on $PU(n_{j+1} + 1) \times \cdots \times PU(n_m + 1)$ factor, is determined by the structure of fibres. Note that a flag manifold is a *GKM manifold* (see [GZ] for details), i.e., the one-skeleton of orbit space of torus action has the structure of a graph; more precisely, zero-dimensional orbits are regarded as vertices and invariant 2-spheres consisting of one-dimensional orbits are regarded as edges in the orbit space. By using this fact and the above equivariant diffeomorphism, we have the following proposition.

Proposition 2.2. The flagified Bott manifold is a GKM manifold.

2.2. **Flagified Bott-Samelson varieties.** There is the Bott-Samelson variety counterpart for the flagified Bott manifolds, called a *flagified Bott-Samelson variety.* Let us briefly introduce it (see the upcoming paper [KLSS] for details). We first choose an element $w_j \in W$ from the Weyl group of G for $j = 1, \dots, n$. Moreover, we assume that each w_j is the longest element of some Weyl subgroup W_j in W, i.e., W_j is generated by some (fixed) generators in W. Then, the following subvariety

$X(w_1, \dots, w_n) \subset (G/B)^n$ is called a *flagified Bott-Samelson variety*:

$$\begin{aligned} &X(w_1, \dots, w_n) \\ &= \{(g_1B, g_2B, \dots, g_nB) \mid g_{j-1}^{-1}g_j \in \overline{Bw_jB}, \text{ for all } j = 1, \dots, n\}, \end{aligned}$$

where $g_0 = e$. If we choose each w_j as the simple reflection s_j, then this is nothing but the Bott-Samelson variety. In the above case, it is known that $X(w_j)$ is the flag manifold whose dimension depends on the length of w_j; for example, $X(s_j)$ is $\mathbb{C}P^1$. Therefore, by iterating the projections from the 1st factor of $X(w_1, \dots, w_n)$, we see that $X(w_1, \dots, w_n)$ is the iterated flag bundle. In particular, we have the following fact:

Proposition 2.3. The flagified Bott-Samelson variety $X(w_1, \dots, w_n)$ is a smooth manifold.

In [KLSS], we show that $X(w_1, \dots, w_n)$ is diffeomorphic to a flagified Bott manifold. However, the natural complex structure induced from $(G/B)^n$ (i.e., regarding as a complex submanifold of $(G/B)^n$) is different from that of the flagified Bott manifold induced from the flagified bundles (i.e., regarding as a $\mathbb{C}P$-tower, see Section 2.3); in [KLSS], we also construct a deformation of complex structures from a flagified Bott-Samelson variety to a flagified Bott manifold.

2.3. **Relations among the generalizations of Bott manifolds.** As a final part of this section, we remark the relations among the generalizations of Bott manifolds. Due to the examples in [KS15, Section 2], the flag manifold may be regarded as a $\mathbb{C}P$-tower. From this observation, we also know that the flagified Bott manifold is also a $\mathbb{C}P$-tower. Moreover, the lowest dimension of flag manifolds is the one-dimensional complex projective space $\mathbb{C}P^1$. This shows that the intersection of the set of flagified Bott manifolds and the set of generalized Bott manifolds is nothing but the set of Bott manifolds. Therefore, there is the relations in Figure 1 among the manifolds which are generalizations of Bott manifolds.

3. Outline of the proof of main theorem

We next give an outline of the proof of Theorem 1.1, see [KLSS] for details. Recall that we can define a free abelian group with finite rank $\mathcal{A}(\Gamma, \alpha, \nabla)$ for the GKM graph (Γ, α, ∇) in the paper [Ku], called a *group of axial functions*. We have the following fact (see [Ku, Corollary 3.1]):

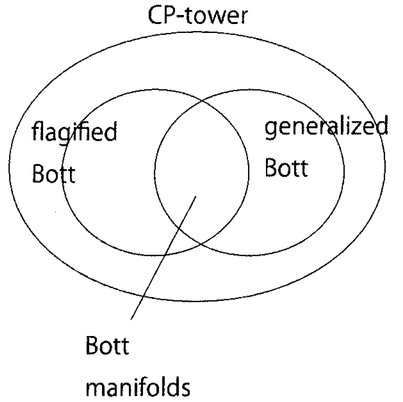

FIGURE 1. The relations among the generalizations of Bott manifolds.

Theorem 3.1. *Let $(\Gamma_M, \alpha_M, \nabla_M)$ be the GKM graph induced from the $2m$-dimensional GKM manifold with n-dimensional torus T^n-action. If* rank $\mathcal{A}(\Gamma_M, \alpha_M, \nabla_M) = n$, *then this T^n-action is the maximal torus action which preserving the almost complex structure.*

Therefore, because the flagified Bott manifold $M = F_m$ defined in Section 2.1 is a $\sum_{j=1}^{m} n_j(n_j+1)$-dimensional GKM manifold with the natural $(n_1 + \cdots + n_m)$-dimensional torus action, it is enough to check that its induced GKM graph $(\Gamma_m, \alpha_m, \nabla_m)$ satisfies that

$$\operatorname{rank} \mathcal{A}(\Gamma_m, \alpha_m, \nabla_m) = n_1 + \cdots + n_m.$$

For the flagified Bott manifold F_m, because its sequence of flag bundles is equivariant, it is easy to show that there is the following sequence of GKM fibrations (see [GSZ] for details) on its induced GKM graph:

$$(\Gamma_m, \alpha_m, \nabla_m) \to (\Gamma_{m-1}, \alpha_{m-1}, \nabla_{m-1}) \to \cdots \to (\Gamma_1, \alpha_1, \nabla_1),$$

where $(\Gamma_1, \alpha_1, \nabla_1)$ is the GKM graph of the $n_1(n_1+1)$-dimensional flag manifold with T^{n_1}-action with the fixed connection, i.e., the connection induced from the invariant complex structure on $SL(n_1+1, \mathbb{C})/B$. In this paper, we only give the outline of the proof for the case of $m = 2$, i.e., 2-stage flagified Bott manifold, because we can easily generalize the proof for this case to the general m-stage flagified Bott manifold.

We first claim the following fact:

Proposition 3.2. The GKM graph $(\Gamma_{(n)}, \alpha_{(n)}, \nabla_{(n)})$ induced from the flag manifold $M = SL(n+1, \mathbb{C})/B$ satisfies that $\mathcal{A}(\Gamma_{(n)}, \alpha_{(n)}, \nabla_{(n)}) \simeq \mathbb{Z}^n$.

Therefore, the T^n-action on the flag manifold $SL(n+1)/B$ is the maximal torus action. Moreover, as a corollary of this proposition, we have that

$$\mathcal{A}(\Gamma_1, \alpha_1, \nabla_1) = \mathcal{A}(\Gamma_{(n_1)}, \alpha_{(n_1)}, \nabla_{(n_1)}) \simeq \mathbb{Z}^{n_1}.$$

We next consider $(\Gamma_2, \alpha_2, \nabla_2)$, the induced GKM graph from F_2, where the 2-stage flagified Bott manifold F_2 satisfies the fibration $SL(n_2+1)/B \to F_2 \to F_1 = SL(n_1+1)/B$. We can check the graph Γ_2 is combinatorially equivalent to $\Gamma_{(n_1)} \times \Gamma_{(n_2)}$. The following fact is the key fact to show the main theorem:

Proposition 3.3. Up to weak equivariant diffeomorphism on F_2, we may change the axial function on $(\Gamma_2, \alpha_2, \nabla_2)$ which satisfies the following properties: there exists two GKM subgraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ in $(\Gamma_2, \alpha_2, \nabla_2)$ such that

- $\mathcal{G}_j = (\Gamma_{(n_j)}, \alpha_{(n_j)}, \nabla_{(n_j)})$;
- the intersection of $\mathcal{G}_1$ and $\mathcal{G}_2$ is the one vertex.

Note that the maximality of the dimension of torus acting on a manifold is not changed by the weak equivariant diffeomorphism, i.e., equivariant diffeomorphism up to automorphism on the torus.

By the computation of the group of axial functions, this proposition shows the following lemma:

Lemma 3.4. rank $\mathcal{A}(\Gamma_2, \alpha_2, \nabla_2) \leq$ rank $\mathcal{A}(\mathcal{G}_1)+$rank $\mathcal{A}(\mathcal{G}_2) = n_1+n_2$.

On the other hand, because there is an (n_1+n_2)-dimensional torus action on F_2, it follows from [Ku, Theorem 3.1] that $n_1+n_2 \leq \mathrm{rank}\mathcal{A}(\Gamma_2, \alpha_2, \nabla_2)$. Together with the above lemma, we establish the following equality:

$$\mathrm{rank}\ \mathcal{A}(\Gamma_2, \alpha_2, \nabla_2) = n_1 + n_2.$$

Consequently, by Theorem 3.1, we have that

Theorem 3.5. *The natural (n_1+n_2)-dimensional torus action on F_2 is the maximal torus action.*

The detailed proofs of the above facts and Theorem 1.1 will be appeared in [KLSS].

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